

HANKEL OPERATORS IN SEVERAL COMPLEX VARIABLES AND  
PRODUCT  $BMO$ 

MICHAEL LACEY AND ERIN TERWILLEGER

ABSTRACT.  $H^2(\otimes_1^n \mathbb{C}_+)$  denotes the Hardy space of square integrable functions analytic in each variable separately. Let  $P^\ominus$  be the natural projection of  $L^2(\otimes_1^n \mathbb{C}_+)$  onto  $\overline{H^2(\otimes_1^n \mathbb{C}_+)}$ . A Hankel operator with symbol  $b$  is the linear operator from  $H^2(\otimes_1^n \mathbb{C}_+)$  to  $\overline{H^2(\otimes_1^n \mathbb{C}_+)}$  given by  $H_b \varphi = P^\ominus \bar{b}\varphi$ . We show that

$$\|H_b\| \simeq \|P^\oplus b\|_{BMO(\otimes_1^n \mathbb{C}_+)},$$

where the right hand norm is S.-Y. Chang and R. Fefferman product  $BMO$ . This fact has well known equivalences in terms of commutators and the weak factorization of  $H^1(\otimes_1^n \mathbb{C}_+)$ . In the case of two complex variables, this is due to Ferguson and Lacey [8]. While the current proof is inductive, and one can take the one complex variable case as the basis step, it is heavily influenced by the methods of Ferguson and Lacey. The induction is carried out with a particular form of a lemma due to Journé [10], which occurs implicitly in the work of J. Pipher [13].

## 1. INTRODUCTION

We characterize the boundedness of Hankel operators in three and more complex variables in terms of the  $BMO$  norm of the symbol of the operator. In one complex variable, this and related facts are the circle of ideas around Nehari's theorem. In the case of more than one complex variable, there are different types of Hankel operators, and we only consider the so called little Hankel operators; little in the sense that the projection used in the definition is onto the smallest natural choice of subspaces of  $L^2(\mathbb{R}^n)$  to use. The structure of these Hankel operators is more intricate due to the more complicated structure of the Hardy spaces  $H^1$  in the product domain and their duals, as identified by S.-Y. Chang and R. Fefferman [[1],[2], [3]]. Some of the tools that have proved to be so flexible and powerful in the one parameter situation apparently have no analog in the higher parameter case; these spaces remain, to a significant degree, poorly understood.

We prove the natural statement about the boundedness of little Hankel operators in an arbitrary number of complex variables. Namely, the Hankel operator with symbol  $b$  is bounded iff the projection of  $b$  into product Hardy space is in product  $BMO$ . Central to this paper is the result of S. Ferguson and M. Lacey [8] that established a similar characterization for Hankel operators of two complex variables. The current proof is inductive in nature, and one can use the classical one variable statements of our theorem as the base case in the induction. In particular the methods of [8] are not sufficient to prove the Theorem in this paper; the inductive argument is the essential new argument in this paper.

Recall that  $L^2(\mathbb{R})$  has the orthogonal decomposition  $H^2(\mathbb{R}) \oplus \overline{H^2(\mathbb{R})}$ . Let  $P^\pm$  be the corresponding orthogonal projections onto the analytic/antianalytic spaces.

---

Received by the editors February 1, 2008.

1991 *Mathematics Subject Classification*. Primary 47B35, 32A35, 32A37. Secondary 42C40.

In  $n$  variables, let  $P_j^\pm$  be the same projections acting on the  $j$ th coordinate,  $j \in \{1, 2, \dots, n\}$ . For functions  $\sigma : \{1, 2, \dots, n\} \longrightarrow \{\pm\}$ , let

$$P^\sigma = \prod_{j=1}^n P_j^{\sigma(j)}.$$

It is clear that  $L^2(\mathbb{R}^n)$  has the orthogonal decomposition into

$$L^2(\mathbb{R}^n) = \oplus_\sigma P^\sigma L^2(\mathbb{R}^n).$$

We take  $\oplus$  to be the function from  $\{1, 2, \dots, n\}$  that is identically  $+$ . It is clear that  $P^\oplus L^2(\mathbb{R}^n) = H^2(\otimes_1^n \mathbb{C}_+)$ .

For a function  $b$ , we set the *Hankel operator with symbol  $b$*  to be  $H_b f = P^\oplus \bar{b} f$ , defined as a map from  $H^2(\otimes_1^n \mathbb{C}_+)$  to  $\overline{H^2(\otimes_1^n \mathbb{C}_+)}$ .  $P^\oplus$  is the projection from  $L^2(\mathbb{R}^n)$  onto  $\overline{H^2(\otimes_1^n \mathbb{C}_+)}$ . Clearly, this operator depends only on  $P^\oplus b$ .

**Theorem 1.1.** *We have the equivalence of norms*

$$(1.2) \quad \|H_b\| \simeq \|P^\oplus b\|_{BMO(\otimes_1^n \mathbb{C}_+)}.$$

Here  $BMO(\otimes_1^n \mathbb{C}_+)$  is the (analytic) Bounded Mean Oscillation space, dual to  $H^1(\otimes_1^n \mathbb{C}_+)$ , as identified by S.-Y. Chang and R. Fefferman. This theorem has two well known equivalences. One is in terms of the commutator

$$C_b := [\cdots [M_b, H_1], H_2], \cdots, H_n],$$

in which  $M_b$  is the operator of pointwise multiplication by  $b$ , and  $H_j$  denotes the Hilbert transform computed in the  $j$ th coordinate. The commutator is a sum of  $2^n$  Hankel operators, each coming from one of the  $2^n$  orthants of  $\mathbb{R}^n$ . In particular, if the *signature* of  $\sigma$  is  $\text{sgn}(\sigma) = \prod_{j=1}^n \sigma(j)$ , a straightforward computation shows that  $C_b = -2^n \sum_\sigma \text{sgn}(\sigma) P^{-\sigma} M_b P^\sigma$ . Thus, the upper bound for the Hankel operators  $\|H_b\| \lesssim \|b\|_{BMO}$  immediately extends to an  $L^2$  operator norm for the commutators. Conversely, assuming the commutator is bounded on  $L^2(\otimes_1^n \mathbb{C}_+)$ , a number of Hankel operators with the same symbol are also bounded. Namely, the Hankel operators are from  $P^\sigma L^2$  to  $P^{-\sigma} L^2$ . Thus the lower bound follows. That is, we have

$$\|C_b\|_2 \simeq \|b\|_{BMO(\otimes_1^n \mathbb{C}_+)}.$$

The latter space is the *real*  $BMO$  space.

A second equivalence is in essence a dual statement to the estimates above, and hence is a statement about  $H^1(\otimes_1^n \mathbb{C}_+)$ . It gives us a *weak factorization* result for that space, namely

$$(1.3) \quad H^1(\otimes_1^n \mathbb{C}_+) = H^2(\otimes_1^n \mathbb{C}_+) \hat{\otimes} H^2(\otimes_1^n \mathbb{C}_+),$$

where the right hand side is the projective tensor product of  $H^2$ . This equality plays a role in our proof, and so we return to it below.

For the proof, the strategy is one of induction on the number of parameters in a manner analogous to the overall strategy of Ferguson and Lacey [8], which addresses the two parameter case. The upper bound on  $\|H_b\|$  in (1.2) is in fact easy to obtain, a fact which is easiest to see via the trivial inclusion in (1.3). Thus, the real difficulty in our theorem lies in the lower bound on  $\|H_b\|$ . Here, there is a bound which follows from the one parameter theory, namely that the operator norm of  $H_b$  is bounded below by the “rectangular  $BMO$ ” norm of  $b$ . It is well known that the rectangular  $BMO$  norm is essentially smaller than the  $BMO$  norm. Ferguson and Lacey [8] showed how to use the Journé Lemma [10] to pass from this essentially smaller norm to the  $BMO$  norm of Chang and Fefferman.

A formulation of the Journé Lemma for rectangles in three and higher parameters is due to J. Pipher [13], but the direct application of this lemma cannot succeed in a proof of our theorem.

The reasons are both technical and heuristic. Relying on just the rectangular  $BMO$  norm in three and more parameters does not take advantage of the subtle way that the  $n$  parameter  $BMO$  space is built up from the  $n - 1$  parameter space.

We find that this point of view, and a form of Journé's Lemma we need, as stated in Section 6, are implicit in the paper of J. Pipher. To use the Journé Lemma, we need to make a definition of  $BMO_{-1}(\otimes_1^n \mathbb{C}_+)$ , which is applied to a function in the  $n$  parameter setting. Our induction argument then, in proving the lower bound in the  $n$  parameter setting, is to first derive the weaker bound of  $BMO_{-1}(\otimes_1^n \mathbb{C}_+)$ . And then prove the correct  $BMO$  bound, assuming that the  $BMO_{-1}(\otimes_1^n \mathbb{C}_+)$  norm of the symbol is sufficiently small.

By  $A \lesssim B$  we mean that there is an absolute constant  $K$  for which  $A \leq KB$ .  $K$  is allowed to depend upon relevant parameters.

We are indebted to J. Pipher for sharing some of her insights into the Journé Lemma, and to the referee for a quick and helpful report.

## 2. THE UPPER BOUND

The upper bound  $\|H_b\|_2 \lesssim \|b\|_{BMO(\otimes_1^n \mathbb{C}_+)}$  can be seen by a soft proof. Consider the Hankel operator  $H_b$ ,

$$(2.1) \quad \begin{aligned} \|H_b\| &= \sup_{\substack{f, g \in H^2(\otimes_1^n \mathbb{C}_+) \\ \|f\|_2=1, \|g\|_2=1}} \int (P^\ominus \bar{b}f)g \, dx \\ &= \sup_{\substack{f, g \in H^2(\otimes_1^n \mathbb{C}_+) \\ \|f\|_2=1, \|g\|_2=1}} \int \overline{P^\oplus b}fg \, dx. \end{aligned}$$

Since the product of  $H^2$  functions is in  $H^1$ , we see that the integral above admits the upper bound of  $\|P^\oplus b\|_{BMO(\otimes_1^n \mathbb{C}_+)}$ . This is the upper half of Theorem 1.1.

We turn to the weak factorization result. For  $A, B$  closed subspaces of  $L^2(\mathbb{R}^n)$ , we define the projective tensor product  $A \hat{\otimes} B \subseteq L^1(\mathbb{R}^n)$  by

$$A \hat{\otimes} B := \left\{ h = \sum_{j=1}^{\infty} f_j g_j \mid (f_j) \subseteq A, (g_j) \subseteq B, \text{ and } \sum_{j=1}^{\infty} \|f_j\|_2 \|g_j\|_2 < \infty \right\}.$$

Observe that that (2.1) implies that the Hankel operator with symbol  $b$  is bounded if and only if the function  $P^\oplus b$  is in the dual of  $H^2(\otimes_1^n \mathbb{C}_+) \hat{\otimes} H^2(\otimes_1^n \mathbb{C}_+)$ . Therefore, the weak factorization equivalence (1.3) is equivalent to our main theorem. That is, we have the equivalence

$$\|H_b\| \simeq \|b\|_{(H^2(\otimes_1^n \mathbb{C}_+) \hat{\otimes} H^2(\otimes_1^n \mathbb{C}_+))^*}.$$

## 3. WAVELETS, $BMO(\otimes_1^n \mathbb{C}_+)$ AND $BMO_{-1}(\otimes_1^n \mathbb{C}_+)$

We begin with some preliminary definitions and calculations in the one parameter setting which carry over naturally to the higher parameter setting. The proofs in the rest of the paper use analytic wavelets constructed by Y. Meyer [11] which are compact in frequency. Let  $w$  be a Schwartz function with  $\|w\|_2 = 1$  and  $\hat{w}(\xi)$  supported on  $[2/3, 8/3]$ . Therefore the wavelets and projections have the nice decay estimates

$$|w(x)| \lesssim (1 + |x|)^{-n} \quad \text{for } n \geq 1.$$

Let  $\mathcal{D}$  denote the dyadic intervals on  $\mathbb{R}$ . For an interval  $I \in \mathcal{D}$ , define

$$w_I(x) := |I|^{-\frac{1}{2}} w\left(\frac{x - c(I)}{|I|}\right),$$

where  $c(I)$  denotes the center of  $I$ . Note that the functions  $w_I(x)$  are well localized to the interval  $I$ . Indeed,

$$|w_I(x)| \lesssim |I|^{-\frac{1}{2}} \left( \frac{1 + \text{dist}(x, I)}{|I|} \right)^{-n} \text{ for } n \geq 1.$$

Y. Meyer has shown that we can choose  $w$  so that  $\{w_I\}_{I \in \mathcal{D}}$  form an orthonormal basis on  $H^2(\otimes_1^n \mathbb{C}_+)$ . Another useful property of these functions is that we have the Littlewood-Paley inequalities,

$$\left\| \sum_I \langle f, w_I \rangle w_I \right\|_p \simeq \left\| \left( \sum_{I \in \mathcal{D}} \frac{|\langle f, w_I \rangle|^2}{|I|} 1_I \right)^{\frac{1}{2}} \right\|_p, \quad 1 < p < \infty.$$

We are now ready to define a characterization of product  $BMO(\otimes_1^n \mathbb{C}_+)$  due to S.-Y. Chang and R. Fefferman [2]. Let  $\mathcal{R} = \mathcal{D}^n = \otimes_{j=1}^n \mathcal{D}$  be the dyadic rectangles. For a rectangle  $R = \otimes_{j=1}^n R_j \in \mathcal{R}$ , define

$$v_R(x) := \prod_{j=1}^n w_{R_j}(x_j).$$

We say  $f \in BMO(\otimes_1^n \mathbb{C}_+)$  iff

$$\sup_U \left[ |U|^{-1} \sum_{R \subset U} |\langle f, v_R \rangle|^2 \right]^{\frac{1}{2}} < \infty$$

where  $U$  is an open set in  $\mathbb{R}^n$  of finite measure.<sup>1</sup> We denote this supremum by  $\|f\|_{BMO(\otimes_1^n \mathbb{C}_+)}$ . It is a theorem of Chang and Fefferman that this definition coincides with the norm of the dual to  $H^1(\otimes_1^n \mathbb{C}_+)$ .

We now define a weaker notion, which we dub  $BMO_{-1}(\otimes_1^n \mathbb{C}_+)$ . For a collection of rectangles  $\mathcal{U} \subset \mathcal{R}$ , set the *shadow* of  $\mathcal{U}$ , to be

$$\text{sh}(\mathcal{U}) := \bigcup_{R \in \mathcal{U}} R.$$

We say that  $\mathcal{U}$  has  $n-1$  parameters iff there is a coordinate  $1 \leq k \leq n$  and a dyadic interval  $I$  so that for all  $R \in \mathcal{U}$ , we have  $R_k = I$ . We then define

$$\|b\|_{BMO_{-1}(\otimes_1^n \mathbb{C}_+)} = \sup_{\mathcal{U}, n-1 \text{ parameters}} \left[ |\text{sh}(\mathcal{U})|^{-1} \sum_{R \in \mathcal{U}} |\langle b, v_R \rangle|^2 \right]^{\frac{1}{2}}.$$

Here, we note that the definition depends only upon the projection  $P^\oplus b$ . In two dimensions, this reduces to a notion that is just slightly weaker than the notion of rectangular  $BMO$ , which is well known to be essentially smaller than the  $BMO$  norm.

#### 4. THE $BMO_{-1}(\otimes_1^n \mathbb{C}_+)$ LOWER BOUND

An essential part of the argument is to use the induction hypothesis to show that we have the lower bound

$$\|H_b\| \gtrsim \|P^\oplus b\|_{BMO_{-1}(\otimes_1^n \mathbb{C}_+)}.$$

This amounts to the assertion that

$$(4.1) \quad \|b\|_{(H^2(\otimes_1^n \mathbb{C}_+) \hat{\otimes} H^2(\otimes_1^n \mathbb{C}_+))^*} \gtrsim \|b\|_{BMO_{-1}(\otimes_1^n \mathbb{C}_+)},$$

an inequality we will demonstrate by relying on the truth of Theorem 1.1 in the  $n-1$  parameter setting.

Given a symbol  $b = b(x_1, x_2, \dots, x_n) = b(x_1, x')$  of  $n$  variables, we assume that  $b$  is analytic in all variables and has  $\|b\|_{BMO_{-1}(\otimes_1^n \mathbb{C}_+)} = 1$ . We also take as given a set  $\mathcal{U}$  of rectangles in  $\mathcal{D}^n$  of

<sup>1</sup>This is analytic  $BMO$ , as we are using analytic wavelets. Real  $BMO$  has a similar definition, provided one uses wavelets that form a basis for  $L^2(\mathbb{R})$ .

$n - 1$  parameters. Thus associated to  $\mathcal{U}$  are the dyadic interval  $I$  and the collection  $\mathcal{U}^{(n-1)} \subset \mathcal{D}^{n-1}$  as in the definition. We assume that  $|I| = 1$ ,  $|\text{sh}(\mathcal{U})| \simeq 1$ , and for all  $R \in \mathcal{U}$  we have  $R_1 = I$  and  $R = I \times R'$ .

Our claim is then that the function

$$\psi = \sum_{R \in \mathcal{U}} \langle b, v_R \rangle v_R$$

has  $H^2(\otimes_1^n \mathbb{C}_+) \widehat{\otimes} H^2(\otimes_1^n \mathbb{C}_+)$  norm  $\lesssim 1$ , which, together with  $\langle \psi, b \rangle = 1$ , certainly proves (4.1). Thus it suffices to show the claim.

Now, since for each  $R \in \mathcal{U}$ , we have  $v_R(x_1, x') = w_I(x_1)v_{R'}(x')$ , and

$$\psi(x_1, x') = w_I(x_1) \sum_{R \in \mathcal{U}} \langle b, v_R \rangle v_{R'}(x') := w_I(x_1)\psi'(x'),$$

we can utilize factorization results in both  $x_1$  and  $x'$ . For  $x_1$ , we use the classical inner outer factorization to conclude that

$$w_I = w_I^{(1)} w_I^{(2)}, \quad \|w_I^{(1)}\|_{H^2(\mathbb{C}_+)} \|w_I^{(2)}\|_{H^2(\mathbb{C}_+)} \lesssim 1.$$

Concerning the function  $\psi'$ , by our choice of  $\mathcal{U}$  and the square function characterization of the Hardy space, we observe that  $\psi'$  has  $H^1(\otimes_{j=1}^{n-1} \mathbb{C}_+)$  norm at most a constant. By the induction hypothesis and, in particular, the assertion that  $H^1(\otimes_{j=1}^{n-1} \mathbb{C}_+) = H^2(\otimes_{j=1}^{n-1} \mathbb{C}_+) \widehat{\otimes} H^2(\otimes_{j=1}^{n-1} \mathbb{C}_+)$ , we can write

$$\psi'(x') = \sum_k \varphi_k(x') \phi_k(x'), \quad \sum_k \|\varphi_k\|_{H^2(\otimes_{j=1}^{n-1} \mathbb{C}_+)} \|\phi_k\|_{H^2(\otimes_{j=1}^{n-1} \mathbb{C}_+)} \lesssim 1.$$

Hence, writing

$$\psi(x_1, x') = \sum_k [w_I^{(1)}(x_1) \varphi_k(x')] \cdot [w_I^{(2)}(x_1) \phi_k(x')]$$

we see that our claim holds. This completes the proof of the lower bound.

## 5. THE $BMO$ LOWER BOUND

An example of Carleson shows that the  $BMO_{-1}(\otimes_1^n \mathbb{C}_+)$  bound is essentially smaller than the  $BMO$  norm. In particular, some tool is needed to pass to the larger norm. That tool is a Journé Lemma, which we detail in the next section.

We can assume that  $\|b\|_{BMO(\otimes_1^n \mathbb{C}_+)} = 1$  and seek an absolute lower bound on  $\|H_b\|$ . In the course of the proof, we will need absolute positive constants  $\delta_{-1}$ ,  $\delta_{\text{journé}}$ ,  $\delta_2$ , and  $\delta_3$ . Other parameters, termed “diagonalization parameters”, are introduced to gain convergent geometric series. The parameters used for these will be denoted with the letter  $d$  with various subscripts. We assume that  $\|b\|_{BMO_{-1}(\otimes_1^n \mathbb{C}_+)} < \delta_{-1}$ , for otherwise we have an absolute lower bound on  $\|H_b\|$ .

Take a set of rectangles  $\mathcal{U}$  which achieves the supremum in the definition of the  $BMO$  norm of  $B$ . We can assume, after a harmless dilation, that  $\frac{1}{2} < |\text{sh}(\mathcal{U})| \leq 1$ . We will show that

$$(5.1) \quad \|H_b \overline{P[\mathcal{U}]b}\|_2 \geq \delta_3.$$

Here we use the notation  $P[\mathcal{U}] = \sum_{R \in \mathcal{U}} v_R \otimes v_R$ , and so  $P[\mathcal{U}]b = \sum_{R \in \mathcal{U}} \langle b, v_R \rangle v_R$ . Establishing the lower bound on the norm of the Hankel operator will require some careful analysis which centers around a variety of paraproducts, proper formulation, and application of a lemma due to Journé which is specified in Section 6.

From the discussion in Section 6, there is a set  $V \supset \text{sh}(\mathcal{U})$ , satisfying several conditions, among them

$$|V| \leq (1 + \delta_{\text{journé}}) |\text{sh}(\mathcal{U})|.$$

Take the collection of rectangles  $\mathcal{V}$  and  $\mathcal{W}$  to be

$$\begin{aligned}\mathcal{V} &:= \{R \in \mathcal{R} \mid R \subset V, R \not\subset \text{sh}(\mathcal{U})\}, \\ \mathcal{W} &:= \mathcal{R} - \mathcal{U} - \mathcal{V}.\end{aligned}$$

We shall prove that for absolute  $\delta_2 > 0$ ,

$$(5.2) \quad \|P^\ominus P[\mathcal{U}]b\overline{P[\mathcal{U}]b}\|_2 \geq \delta_2,$$

$$(5.3) \quad \|P^\ominus P[\mathcal{V}]b\overline{P[\mathcal{U}]b}\|_2 \lesssim \delta_{\text{journé}}^{1/2},$$

$$(5.4) \quad \|P^\ominus P[\mathcal{W}]b\overline{P[\mathcal{U}]b}\|_2 \leq K_{\delta_{\text{journé}}} \delta_{-1}.$$

The last inequality holds with a constant that depends only on  $\delta_{\text{journé}}$ . Thus, fixing first  $\delta_{\text{journé}}$  sufficiently small and then  $\delta_{-1}$  proves (5.1).

The first two estimates are trivial, as we indicate now. First, note that the Fourier transform of  $|P[\mathcal{U}]b|^2$  is symmetric, so that

$$\|P^\ominus P[\mathcal{U}]b\overline{P[\mathcal{U}]b}\|_2 \geq 2^{-n} \|P[\mathcal{U}]b\|_4^2.$$

The  $L^4$  norm has a lower bound, due to the fact that we have taken the shadow of  $\mathcal{U}$  to have measure approximately one and the validity of the Littlewood-Paley inequalities. Thus,

$$\begin{aligned}\frac{1}{4} &\leq \|P[\mathcal{U}]b\|_2 \\ &= \sum_{R \in \mathcal{U}} |\langle b, v_R \rangle|^2 \\ &\leq \left\| \left[ \sum_{R \in \mathcal{U}} \frac{|\langle b, v_R \rangle|^2}{|R|} 1_R \right]^{1/2} \right\|_4 \\ &\lesssim \|P[\mathcal{U}]b\|_4.\end{aligned}$$

This proves (5.2).

Second, use the control on the size of  $V$  to see that

$$\|P[\mathcal{U}]b\|_2^2 + \|P[\mathcal{V}]b\|_2^2 = |\text{sh}(\mathcal{U})| + \|P[\mathcal{V}]b\|_2^2 \leq |V| \leq (1 + \delta_{\text{journé}}) |\text{sh}(\mathcal{U})|.$$

Thus,  $\|P[\mathcal{V}]b\|_2^2 \leq \delta_{\text{journé}}$ . By the John-Nirenberg inequality, we see that  $\|P[\mathcal{V}]b\|_4 \leq \delta_{\text{journé}}^{1/4}$ . Hence, we can prove (5.3) as follows.

$$\|P^\ominus P[\mathcal{V}]b\overline{P[\mathcal{U}]b}\|_2 \leq \|P[\mathcal{U}]b\|_4 \|P[\mathcal{V}]b\|_4 \lesssim \delta_{\text{journé}}^{1/2}.$$

**The Definitions of the Paraproducts.** The principal inequality is (5.4), and it requires a sustained analysis to verify. It is imperative to observe that the term  $H_{P[\mathcal{W}]b} P[\mathcal{U}]b$  has a sizable cancellation as a sum over wavelets. If  $R$  and  $R'$  are two dyadic rectangles with  $|R_j| < 8|R'_j|$  for any  $1 \leq j \leq n$ , then we would have

$$P^\ominus v_{R'} \overline{v_R} = 0.$$

This is due to the fact that in the  $j$ th coordinate, the Fourier transform is not supported in  $\xi_j < 0$ . Thus, we can replace the definition of  $\mathcal{W}$  by:

$$\mathcal{W} = \{R' \in \mathcal{R} \mid R' \not\subset V, \text{ and for some } R \in \mathcal{U}, |R'_j| < 8|R_j| \text{ for all } 1 \leq j \leq n\}.$$

It is also imperative to observe that even with this restricted definition, the shadows of  $\mathcal{U}$  and  $\mathcal{W}$  will, in general, overlap. This overlap will be controlled by the Journé Lemma and additional orthogonality considerations.

Nevertheless, the sum should be analyzed along the lines of a product of two functions which are nearly supported on disjoint sets. The technique for doing this is via sums known generically as paraproducts. In  $n$  parameters, the paraproducts admit different degeneracies, as measured in

the amount of orthogonality present in the sums. It is the purpose of the following definitions to quantify these paraproducts.

Given a subset  $J \subset \{1, 2, \dots, n\}$ , write  $R' \prec_J R$  iff for indices  $j \in J$ , we have  $8|R'_j| < |R_j|$ , whereas for indices  $j \in \{1, 2, \dots, n\} - J$ , we have  $8^{-1}|R'_j| \leq |R_j| \leq 8|R'_j|$ . Set

$$\begin{aligned}\mathcal{X}(J) &:= \{(R', R) \in \mathcal{W} \times \mathcal{U} \mid R' \prec_J R\} \\ \mathbb{X}(J) &:= \sum_{(R', R) \in \mathcal{X}(J)} \overline{\langle b, v_{R'} \rangle v_{R'}} \langle b, v_R \rangle v_R.\end{aligned}$$

The remainder of the proof is devoted to the assertion that

$$(5.5) \quad \|\mathbb{X}(J)\|_2 \leq K_{\delta_{\text{journé}}} \delta_{-1}, \quad J \subset \{1, 2, \dots, n\}.$$

This objective can only be met with additional diagonalizations of the sums. Applying the Journé Lemma as stated in Lemma 6.10, we can decompose  $\mathcal{U}$  into collections  $\mathcal{U}_{d_1}$ , for  $d_1 \in \mathbb{N}$ , for which we have  $2^{d_1} R \subset V$  for  $R \in \mathcal{U}_{d_1}$ , and

$$(5.6) \quad \|\mathbb{P}[\mathcal{U}_{d_1}]b\|_{BMO} \leq K_{\delta_{\text{journé}}} 2^{(n+1)d_1} \|b\|_{BMO_{-1}(\otimes_1^n \mathbb{C}_+)} \lesssim 2^{(n+1)d_1} \delta_{-1}.$$

In what follows, we shall suppress the dependence of these inequalities on the choice of  $\delta_{\text{journé}}$ , which comes only through this application of Journé's Lemma. Also the (large) power of  $d_1$  is of no particular consequence. From another part of the estimate we can pick up a factor of  $K_N 2^{-Nd_1}$  for arbitrarily large  $N$ .

For integers  $d_2 \geq d_1$ , set

$$\begin{aligned}\mathcal{X}(J, d_2) &:= \{(R', R) \in \mathcal{W} \times \mathcal{U}_{d_1} \mid R' \prec_J R, R' \subset 2^{d_2+4} R, R' \not\subset 2^{d_2} R\} \\ \mathbb{X}(J, d_2) &:= \sum_{(R', R) \in \mathcal{X}(J, d_2)} \overline{\langle b, v_{R'} \rangle v_{R'}} \langle b, v_R \rangle v_R.\end{aligned}$$

In this notation, and below, we will suppress the dependence upon  $d_1$ , as this parameter does not directly enter into any of the estimates. We shall show that

$$(5.7) \quad \|\mathbb{X}(J, d_2)\|_2 \lesssim 2^{-d_2} \delta_{-1}, \quad J \subset \{1, 2, \dots, n\}, \quad 0 \leq d_1 \leq d_2.$$

This estimate proves (5.5).

Orthogonality enters into the estimate in the following way. Suppose we are given two pairs of rectangles  $(R, R')$  and  $(\tilde{R}, \tilde{R}')$  in  $\mathcal{X}(J)$ . In addition, suppose  $16|R'_j| < |\tilde{R}'_j|$  for some  $j \in J$ . We conclude that the functions  $v_{R'} \overline{v_R}$  and  $v_{\tilde{R}'} \overline{v_{\tilde{R}}}$  are orthogonal. This is seen by examining the Fourier supports of the wavelets. Therefore, for  $|J|$ -tuples of integers  $\vec{\ell} \in \mathbb{Z}^{|J|}$ , we define

$$\begin{aligned}\mathcal{X}(J, d_2, \vec{\ell}) &:= \{(R', R) \in \mathcal{X}(J, d_2) \mid |R'_j| = 2^{\vec{\ell}_j}, j \in J\}. \\ \mathbb{X}(J, d_2, \vec{\ell}) &:= \sum_{(R', R) \in \mathcal{X}(J, d_2, \vec{\ell})} \overline{\langle b, v_{R'} \rangle v_{R'}} \langle b, v_R \rangle v_R.\end{aligned}$$

Here, for simplicity, we have assumed that  $J = \{1, 2, \dots, |J|\}$  for notational convenience. We will continue with this assumption throughout. All estimates will clearly be invariant under appropriate permutation of coordinates. In light of the orthogonality above, it is the case that

$$(5.8) \quad \|\mathbb{X}(J, d_2)\|_2^2 \lesssim \sum_{\vec{\ell} \in \mathbb{Z}^{|J|}} \|\mathbb{X}(J, d_2, \vec{\ell})\|_2^2.$$

At this point, we can abandon orthogonality considerations altogether in estimating this last sum. In the sums  $\mathbb{X}(J, d_2, \vec{\ell})$  we have the functions  $v_{R'} \overline{v_R}$ , which can be dominated as

$$\begin{aligned}||R'| |R|^{1/2} |v_{R'} \overline{v_R}| &\lesssim [(\zeta_{R'} * 1_{R'}) (\zeta_R * 1_R)]^2 \\ &\lesssim M 1_R (c(R'))^N \zeta_{R'} * 1_{R'},\end{aligned}$$

where we take

$$\zeta_R(x) := [1 + |x_1||R_1|^{-1} + \cdots + |x_n||R_n|^{-1}]^{-100n}.$$

Here,  $N > 1$  is arbitrary, though the implied constant will depend upon the choice of  $N$ .  $M 1_R(c(R'))$  is an effective measure of the distance between  $R'$  and  $R$ , as  $R'$  will always have dimensions which are smaller or comparable to those of  $R$ .

Of course, for  $(R', R) \in \mathcal{X}(J, d_2)$ , we have  $M 1_R(c(R')) \lesssim 2^{-Nd_2}$ . Thus,

$$\begin{aligned} \|\mathbb{X}(J, d_2, \vec{\ell})\|_2 &\lesssim 2^{-Nd_2} \|\tilde{\mathbb{X}}(J, d_2, \vec{\ell})\|_2, \quad \text{where} \\ \tilde{\mathbb{X}}(J, d_2, \vec{\ell}) &:= \sum_{(R', R) \in \mathcal{X}(J, d_2, \vec{\ell})} \frac{\beta(R')}{\sqrt{|R'|}} \frac{\beta(R)}{\sqrt{|R|}} 1_{R'}, \quad \text{and} \\ \beta(R) &:= |\langle b, v_R \rangle|. \end{aligned}$$

The top line holds for all large integers  $N$ . Thus in the argument below we can accrue some bounded number of positive powers of  $2^{d_2}$  and not place our desired estimate in jeopardy. Hence, to obtain (5.7) it is enough for us to show that

$$(5.9) \quad \sum_{\vec{\ell} \in \mathbb{Z}^{|J|}} \|\tilde{\mathbb{X}}(J, d_2, \vec{\ell})\|_2^2 \lesssim 2^{8nd_2} \delta_{-1}^2, \quad J \subset \{1, 2, \dots, n\}.$$

As the terms  $\tilde{\mathbb{X}}$  are sums of indicator sets of rectangles, we can appeal to facts about Carleson measures and, in particular, the John-Nirenberg inequalities to control these sums.

There is a final diagonalization to make. For  $|J|$ -tuples of natural numbers  $\vec{d}_3 \in \mathbb{N}^{|J|}$ , set

$$\begin{aligned} \mathcal{X}(J, d_2, \vec{\ell}, \vec{d}_3) &:= \{(R', R) \in \mathcal{X}(J, d_2, \vec{\ell}) \mid 2^{\vec{d}_{3j}} |R'_j| = |R_j|, j \in J\}, \\ \tilde{\mathbb{X}}(J, d_2, \vec{\ell}, \vec{d}_3) &:= \sum_{(R', R) \in \mathcal{X}(J, d_2, \vec{\ell}, \vec{d}_3)} \frac{\beta(R')}{\sqrt{|R'|}} \frac{\beta(R)}{\sqrt{|R|}} 1_{R'} \\ &= 2^{-\frac{1}{2}\|\vec{d}_3\|} \sum_{(R', R) \in \mathcal{X}(J, d_2, \vec{\ell}, \vec{d}_3)} \frac{\beta(R')\beta(R)}{|R'|} 1_{R'}, \end{aligned}$$

where  $\|\vec{d}_3\| := \sum_{j=1}^{|J|} |\vec{d}_{3j}|$ . The leading term of the last line suggests that indeed  $\vec{d}_3$  is a diagonalization parameter.

With the relative sizes of  $R'$  and  $R$  fixed by the choice of  $J \subset \{1, 2, \dots, n\}$  and by the choice of  $\vec{d}_3$ , observe that for each  $R'$ , there can be at most  $O(2^{nd_2})$  possible choices of  $R$  so that  $(R, R') \in \mathcal{X}(J, d_2, \vec{\ell}, \vec{d}_3)$ . Again, we can afford to lose some bounded number of powers of  $2^{d_2}$  in our estimates. We take

$$\begin{aligned} \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3) &:= \{R' \mid (R', R) \in \mathcal{X}(J, d_2, \vec{\ell}, \vec{d}_3) \text{ for some } R \in \mathcal{R}\}, \\ \mathcal{Y}(J, d_2, \vec{d}_3) &:= \bigcup_{\vec{\ell} \in \mathbb{Z}^{|J|}} \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3). \end{aligned}$$

Let  $\pi : \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3) \longrightarrow \mathcal{R}$  be such that  $(R', \pi(R')) \in \mathcal{X}(J, d_2, \vec{\ell}, \vec{d}_3)$ . We would need to consider  $O(2^{nd_2})$  possible choices for this function. Below, we will consider just some arbitrary choice of this function  $\pi$ , and then merely sum over the possible choices of  $\pi$ , accruing a harmless term of  $O(2^{nd_2})$ .



Set

$$\begin{aligned}
 \mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3) &:= \sum_{R' \in \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3)} \frac{\beta(R')}{\sqrt{|R'|}} \frac{\beta(\pi(R'))}{\sqrt{|\pi(R')|}} 1_{R'} \\
 (5.10) \qquad &= 2^{-\frac{1}{2}\|\vec{d}_3\|} \sum_{R' \in \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3)} \frac{\beta(R')\beta(\pi(R'))}{|R'|} 1_{R'}.
 \end{aligned}$$

The specific estimate we prove is:

$$\begin{aligned}
 \|\mathbb{Y}(J, d_2, \vec{d}_3)\|_2^2 &:= \sum_{\vec{\ell} \in \mathbb{Z}^{|J|}} \|\mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3)\|_2^2 \\
 (5.11) \qquad &\lesssim 2^{8nd_2 - \frac{1}{4}\|\vec{d}_3\|} \delta_{-1}^2, \quad J \subset \{1, 2, \dots, n\}.
 \end{aligned}$$

This is summed over  $\vec{d}_3 \in \mathbb{N}^{|J|}$  to prove (5.9), and so will complete our proof. The proof of this inequality is taken up in the next subsection. We achieve an exponential decay in parameters  $d_1, d_2, \vec{d}_3$ .

We shall rely repeatedly on the estimates

$$(5.12) \qquad \sum_{R \in \mathcal{U}_{d_1}} \beta(R)^2 \lesssim 2^{2(n+1)d_1} \delta_{-1}^2,$$

$$(5.13) \qquad \sum_{R' \in \mathcal{Y}(J, d_2, \vec{d}_3)} \beta(R')^2 \lesssim 2^{2nd_2}, \quad J \subset \{1, 2, \dots, n\}, \vec{d}_3 \in \mathbb{N}^{|J|}.$$

The first of these has the critical gain by a factor of  $\delta_{-1}^2$ , as follows from (5.6). The second estimate follows from the fact that  $b$  is in  $BMO$  and that the rectangles  $R'$  in  $\mathcal{Y}(J, d_2, \vec{d}_3)$  are contained in  $\{M 1_{\text{sh}(\mathcal{U})} > c2^{-nd_2}\}$  since  $R' \subset 2^{d_2+4}R$  for some  $R \in \mathcal{U}_{d_1}$ . Here  $M$  is the strong maximal function.

At this point we recap the notations.

- $\delta_{-1} = \|b\|_{BMO_{-1}(\otimes_1^n \mathbb{C}_+)}$ .
- $d_1$  is associated to the measure of embeddedness of rectangles  $R \in \mathcal{U}$ .
- $d_2$  is a (crude) measure of the separation between the rectangles  $R' \in \mathcal{W}$  and  $R \in \mathcal{U}_{d_1}$  for  $d_2 \geq d_1$ .
- $J \subset \{1, 2, \dots, n\}$  is that set of coordinates for which one has some orthogonality.
- $\vec{\ell} \in \mathbb{Z}^{|J|}$  specifies the side lengths of  $R'$  for those coordinates  $j \in J$ .
- $\vec{d}_3 \in \mathbb{N}^{|J|}$  specifies how much bigger  $R$  is than  $R'$  in the coordinates  $j \in J$ .
- $(R', \pi(R')) \in \mathcal{X}(J, d_2, \vec{\ell}, \vec{d}_3)$  and  $|\pi(R')_j| = 2^{\vec{d}_{3j}} |R'_j|$  for  $j \in J$ , otherwise for  $j \notin J$ ,  $|\pi(R')_j| \simeq |R'_j|$ .
- $\beta(R) := |\langle b, v_R \rangle|$ .

**The Bounds for the Paraproducts.** The argument varies depending upon the cardinality of  $J \subset \{1, 2, \dots, n\}$ . While we can formalize issues in a way that is uniform with respect to  $|J|$ , we present four subsections, to emphasize the differences that come about due to the increasing number of parameters.

*The Case of  $J = \{1, 2, \dots, n\}$ .* The key point is that the sum in (5.11) simplifies considerably, as all the side lengths of  $R'$  are specified by the parameter  $\vec{\ell}$ . In particular, the rectangles  $R'$  occurring

in the sum in (5.10) are pairwise disjoint. Thus, the  $L^2$  norm in (5.11) will simplify to

$$\begin{aligned} \|\mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3)\|_2^2 &= \sum_{R' \in \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3)} \frac{\beta(R')^2 \beta(\pi(R'))^2}{\pi(|R'|)} \\ &= 2^{-\|\vec{d}_3\|} \sum_{R' \in \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3)} \frac{\beta(R')^2 \beta(\pi(R'))^2}{|R'|} \\ &\leq 2^{-\|\vec{d}_3\|} \sup_{R'} \frac{\beta(R')^2}{|R'|} \sum_{R' \in \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3)} \beta(\pi(R'))^2. \end{aligned}$$

As  $b$  has  $BMO$  norm one, the supremum above is bounded by 1. Then sum over  $\vec{\ell} \in \mathbb{Z}^{|J|}$  and use (5.12) to see that

$$\|\mathbb{Y}(J, d_2, \vec{d}_3)\|_2^2 \lesssim \delta_{-1}^2 2^{3nd_2 - \|\vec{d}_3\|}.$$

Recall that we can tolerate a few positive powers of  $d_2$ . This case is complete.

*The Case of  $J = \{1, 2, \dots, n-1\}$ .* Now, the rectangles  $R' \in \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3)$  are only permitted to vary in the last coordinate. That is, the corresponding sums are as complex as those of one parameter Carleson measures. So we can explicitly compute

$$\begin{aligned} \|\mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3)\|_2^2 &= 2^{-\|\vec{d}_3\|} \sum_{R' \in \mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3)} \frac{\beta(R') \beta(\pi(R'))}{|R'|} \\ &\quad \times \sum_{\substack{R'' \in \mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3) \\ R'' \subset R'}} \beta(R'') \beta(\pi(R'')). \end{aligned}$$

With the specific way the innermost sum is formed, observe that

$$\begin{aligned} \sum_{\substack{R'' \in \mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3) \\ R'' \subset R'}} \beta(R'')^2 &\lesssim \delta_{-1}^2 |R'|, \\ \sum_{\substack{R'' \in \mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3) \\ R'' \subset R'}} \beta(\pi(R''))^2 &\lesssim 2^{nd_2 + \|\vec{d}_3\|} \delta_{-1}^2 |R'|. \end{aligned}$$

The first estimate is obvious, while the second estimate follows from the fact that the rectangles  $\pi(R')$  are contained in

$$\otimes_{j=1}^n 2^{d_2 + \vec{d}_{3j}} R'_j.$$

In this last display, set the last coordinate of  $\vec{d}_3$  to be zero. Applying these observations, Cauchy-Schwarz, (5.13), and  $\|b\|_{BMO(\otimes_1^r \mathbb{C}_+)} = 1$  we see that

$$\begin{aligned} \|\mathbb{Y}(J, d_2, \vec{d}_3)\|_2^2 &\lesssim \delta_{-1}^2 2^{-\frac{1}{2}\|\vec{d}_3\|} 2^{\frac{nd_2}{2}} \sum_{R' \in \mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3)} \beta(R') \beta(\pi(R')) \\ &\lesssim \delta_{-1}^2 2^{\frac{5}{2}nd_2 - \frac{1}{2}\|\vec{d}_3\|}. \end{aligned}$$

This completes this case.

*The Case of  $0 < |J| < n - 1$ .* The argument in this case could be adapted to treat the general case. We would like to indicate the additional difficulty that one faces in this case. The side lengths of  $R'$  are fixed for those coordinates in  $J$ , and completely specified by  $\vec{\ell} \in \mathbb{Z}^{|J|}$ . The remaining side lengths of  $R'$  are then permitted to vary. Thus, the ways that two possible choices of  $R', R'' \in \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3)$  can intersect are as general as the intersections of two dyadic rectangles of dimension  $n - |J|$ .

Nevertheless, one can implement a method of proof that follows the lines of the case  $J = \{1, 2, \dots, n-1\}$ , provided one takes advantage of the John-Nirenberg inequality, which we now state in the form used. For rectangles  $R \in \mathcal{R}$  and non-negative constants  $a_R$  for which  $\sum_{R \subset W} a_R \leq |W|$  for all open sets  $W \subset \mathbb{R}^n$ , we have

$$\left\| \sum_{R \subset W} \frac{a_R}{|R|} 1_R \right\|_p \lesssim |W|^{1/p}, \quad 1 < p < \infty.$$

We use this to obtain the following extensions of the inequalities (5.12) and (5.13). In the first place, we have

$$(5.14) \quad \left\| \left[ \sum_{R' \in \mathcal{Y}(J, d_2, \vec{d}_3)} \frac{\beta(R')^2}{|R'|} 1_{R'} \right]^{1/2} \right\|_p \lesssim 2^{2nd_2}, \quad 1 < p < \infty.$$

This is available to us from the fact that  $b$  is in  $BMO$  with norm one. A similar fact is

$$(5.15) \quad \left\| \left[ \sum_{R' \in \mathcal{Y}(J, d_2, \vec{d}_3)} \frac{\beta(\pi(R'))^2}{|\pi(R')|} 1_{R'} \right]^{1/2} \right\|_p \lesssim \delta_{-1} 2^{2nd_2}, \quad 1 < p < \infty.$$

The important features of these estimates are that they are independent of  $\vec{d}_3$ , uniform in  $\vec{\ell} \in \mathbb{Z}^{|J|}$ , and in the second estimate we have the gain of  $\delta_{-1}$ .

Now, the second estimate does not immediately follow from a  $BMO$  estimate, due to the fact that we have a mismatch between  $R'$  and  $\pi(R')$  in (5.15). Due to the John-Nirenberg inequality, (5.15) will follow from the estimate

$$\sum_{\substack{R' \subset W \\ R' \in \mathcal{Y}(J, d_2, \vec{d}_3)}} \beta(\pi(R'))^2 \lesssim 2^{4nd_2 + \|\vec{d}_3\|} \delta_{-1}^2 |W|, \quad W \subset \mathbb{R}^n.$$

As this estimate is uniform in the choice of  $W$ , it provides a bound for a Carleson measure to which the John-Nirenberg inequality applies. For a given  $W \subset \mathbb{R}^n$ , it is the case that for all rectangles  $R'$  that contribute to this sum, the rectangle  $\pi(R')$  is contained in a set which is given in the first place by a strong maximal function applied to  $W$ . Set

$$W_0 := \{M 1_W \geq c 2^{-nd_2}\},$$

for an appropriate choice of  $c$ . This set, so constructed, will contain a translation of  $R'$  which is contained in  $\pi(R')$ . The point to keep in mind is that  $\pi(R')$  is  $2^{\vec{d}_{3j}}$  times longer than  $R'$  in the coordinate  $j \in J$ . Thus, in that coordinate, we should apply a one dimensional maximal function with threshold  $2^{-\vec{d}_{3j}}$ . Namely, for  $j \in J = \{1, 2, \dots, |J|\}$ , we inductively define

$$W_j := \{M_j 1_{W_{j-1}} > c 2^{-\vec{d}_{3j}}\}.$$

For appropriate constant  $c$ , we will have  $\pi(R') \subset W_{|J|}$ . And we certainly have  $|W_{|J|}| \lesssim 2^{\|\vec{d}_3\| + 2nd_2} |W|$ . This completes the proof of (5.15).

Estimates (5.14) and (5.15) are not in themselves enough to complete the proof, as there is no decay in the quantity  $\|\vec{d}_3\|$ . But, they do show that

$$\begin{aligned}
 (5.16) \quad & \left\| \left[ \sum_{\vec{\ell} \in \mathbb{Z}^{|J|}} \mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3)^2 \right]^{1/2} \right\|_4 \\
 & \lesssim \left\| \left[ \sum_{R' \in \mathcal{Y}(J, d_2, \vec{d}_3)} \frac{\beta(R')^2}{|R'|} 1_{R'} \right]^{1/2} \right\|_8 \left\| \left[ \sum_{R' \in \mathcal{Y}(J, d_2, \vec{d}_3)} \frac{\beta(\pi(R'))^2}{|\pi(R')|} 1_{R'} \right]^{1/2} \right\|_8 \\
 & \lesssim 2^{4nd_2} \delta_{-1}.
 \end{aligned}$$

Namely, we have an estimate on the  $L^4$  norm that is uniform with respect to  $\vec{d}_3$ . This will permit us to select a set which decays with respect to this parameter. On this set, we will not attempt to estimate the  $L^2$  norm in (5.11). The set we take is

$$E := \bigcup_{\vec{\ell} \in \mathbb{Z}^{|J|}} \{M\mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3) > \delta_{-1} 2^{\frac{1}{8}\|\vec{d}_3\|}\}.$$

Here, we use the strong maximal function  $M$ . This set has measure  $|E| \lesssim 2^{16nd_2 - \frac{1}{2}\|\vec{d}_3\|}$ , due to the large  $L^p$  norms we have in (5.16).

To complete the argument in this case, it suffices to show that

$$\sum_{\vec{\ell} \in \mathbb{Z}^{|J|}} \int_{\mathbb{R}^n - E} |\mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3)|^2 dx \lesssim \delta_{-1} 2^{3nd_2 - \frac{1}{4}\|\vec{d}_3\|}.$$

We will expand the square on the left hand side. Integrating  $\mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3)$  over  $R' - E$ , we will lose a factor of  $2^{\frac{1}{8}\|\vec{d}_3\|}$ . But from  $|\pi(R')| = 2^{\|\vec{d}_3\|} |R'|$  we will gain a factor of  $2^{\frac{1}{2}\|\vec{d}_3\|}$ . Specifically,

$$\begin{aligned}
 \sum_{\vec{\ell} \in \mathbb{Z}^{|J|}} \int_{\mathbb{R}^n - E} |\mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3)|^2 dx & \leq \sum_{\vec{\ell} \in \mathbb{Z}^{|J|}} \sum_{R' \in \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3)} \frac{\beta(R')}{\sqrt{|R'|}} \frac{\beta(\pi(R'))}{\sqrt{|\pi(R')|}} \times \\
 & \quad \int_{R' - E} \mathbb{Y}(J, d_2, \vec{\ell}, \vec{d}_3) dx \\
 & \leq \delta_{-1} 2^{(\frac{1}{8} - \frac{1}{2})\|\vec{d}_3\|} \sum_{\vec{\ell} \in \mathbb{Z}^{|J|}} \sum_{R' \in \mathcal{Y}(J, d_2, \vec{\ell}, \vec{d}_3)} \beta(R') \beta(\pi(R')) \\
 & \lesssim \delta_{-1}^2 2^{(2n+1)d_2 - \frac{1}{4}\|\vec{d}_3\|}.
 \end{aligned}$$

This estimate follows from the definition of the set  $E$  and (5.12) and (5.13).

*The Case of  $J = \emptyset$ .* In this case, both  $\vec{\ell}$  and  $\vec{d}_3$  are not present, and the rectangles  $R'$  and  $\pi(R')$  have comparable lengths in all coordinates. But we do have (5.14) and (5.15), and they directly prove the desired estimate

$$\begin{aligned}
 \|\mathbb{Y}(\emptyset, d_2)\|_2 & \lesssim \left\| \left[ \sum_{R' \in \mathcal{Y}(\emptyset, d_2)} \frac{\beta(R')^2}{|R'|} 1_{R'} \right]^{1/2} \right\|_4 \left\| \left[ \sum_{R' \in \mathcal{Y}(\emptyset, d_2)} \frac{\beta(\pi(R'))^2}{|\pi(R')|} 1_{R'} \right]^{1/2} \right\|_4 \\
 & \lesssim \delta_{-1} 2^{4nd_2}.
 \end{aligned}$$

And this completes this case.

## 6. JOURNÉ'S LEMMA

We state a version of the Lemma of Journé [10] that is implicit in Pipher's extension [13], and interfaces well with our notion of a restricted  $BMO$  norm, namely  $BMO_{-1}(\otimes_1^n \mathbb{C}_+)$ . We first state the lemma in a purely geometric fashion, and then return to a formulation that is more specific to our needs in this paper.

**The Geometric Formulation.** Given a collection  $\mathcal{U}$  of dyadic rectangles whose shadow has finite area, suppose that  $V \supset \text{sh}(\mathcal{U})$ . For rectangles  $R \in \mathcal{U}$ , define

$$\text{emb}(R, V) := \sup\{\mu \geq 1 \mid \mu R_1 \times R_2 \times \cdots \times R_n \subset V\}.$$

For an arbitrary subset  $\mathcal{U}' \subset \mathcal{U}$ , let

$$F(I, k, \mathcal{U}') := \bigcup \{I \times R' \mid I \times R' \in \mathcal{U}', 2^{k-1} \leq \text{emb}(I \times R', V) < 2^k\}.$$

**Lemma 6.1.** *For all  $\delta, \epsilon > 0$ , we can select  $V \supset \text{sh}(\mathcal{U})$  with  $|V| \leq (1 + \delta)|\text{sh}(\mathcal{U})|$ , for which we have the uniform estimate<sup>2</sup>*

$$(6.2) \quad \sum_{k=1}^{\infty} \sum_{I \in \mathcal{D}} 2^{-\epsilon k} |F(I, k, \mathcal{U}')| \lesssim |\text{sh}(\mathcal{U}')|, \quad \mathcal{U}' \subset \mathcal{U}.$$

*The implied constants in these estimates depend only on dimension and the choices of  $\epsilon, \delta$ .*

Consider a collection of rectangles  $\mathcal{U}$  in which all the first coordinates are the same. Then the embeddedness is necessarily of order 1. This example shows that the lemma above must be formulated in this fashion.

We begin the proof with a careful description of how to select the set  $V$ . If we were not too concerned about the upper bound on the measure of  $V$ , in other words if the bound  $|V| \lesssim |\text{sh}(\mathcal{U})|$  were enough, then we could simply take  $V = \{M 1_{\text{sh}(\mathcal{U})} > \frac{1}{2}\}$ . For our needs, however, this choice of  $V$  is completely inappropriate.

We need the notion of *shifted dyadic grids*, which is a modification of an observation due to M. Christ defined as follows. The definition of the grids depends upon a choice of integer  $d$ , and we will set  $\delta = (2^d + 1)^{-1}$ . For integers  $0 \leq b < d$ , and  $\alpha \in \{\pm(2^d + 1)^{-1}\}$ , let

$$(6.3) \quad \mathcal{D}_{d,b,\alpha} := \{2^{kd+b}((0,1) + j + (-1)^k \alpha) \mid k \in \mathbb{Z}, j \in \mathbb{Z}\},$$

$$\mathcal{D}_d := \bigcup_{\alpha} \bigcup_{b=0}^{d-1} \mathcal{D}_{d,b,\alpha}.$$

One checks that  $\mathcal{D}_{d,b,\alpha}$  is a grid. Indeed, it suffices to assume  $\alpha = (2^d + 1)^{-1}$  and that  $b = 0$ . Checking the grid structure can be done by induction. And it suffices to check that the intervals in  $\mathcal{D}_{d,0,\alpha}$  of length one are a union of intervals in  $\mathcal{D}_{d,0,\alpha}$  of length  $2^{-d}$ . One need only check this for the interval  $(0,1) + \alpha$ . But certainly

$$\begin{aligned} (0,1) + \frac{1}{(2^d + 1)} &= \bigcup_{j=0}^{2^d-1} (0, 2^{-d}) + \frac{j}{2^d} + \frac{1}{(2^d + 1)} \\ &= \bigcup_{j=0}^{2^d-1} (0, 2^{-d}) + \frac{j+1}{2^d} - \frac{1}{2^d(2^d + 1)}. \end{aligned}$$

---

<sup>2</sup>We have stated the lemma in the formulation for the first coordinate to ease the burden of notation. In application, we will use this in an arbitrary choice of coordinate.

What is more important concerns the collections  $\mathcal{D}_d$ . For each dyadic interval  $I \in \mathcal{D}$ ,  $I \pm \delta|I| \in \mathcal{D}_d$ .<sup>3</sup> Moreover, the maximal function  $M^{\mathcal{D}_d}$  maps  $L^1$  into  $L^{1,\infty}$ <sup>4</sup> with norm at most  $2d \simeq |\log \delta|$ . In fact we need the finer estimate, valid for all choices of  $0 < \delta < 1$  and integers  $d$ ,

$$|\{M^{\mathcal{D}_d} 1_U > 1 - \delta\}| \leq (1 + K\delta d)|U|$$

for all subsets  $U$  of the real line of finite measure and some constant  $K$ . This will be an effective estimate since the value of  $d$  we will consider is  $d \simeq \lceil -\log_2 \delta \rceil$ . To see this estimate, note that

$$\begin{aligned} |\{M^{\mathcal{D}_d} 1_U > 1 - \delta\}| &\leq |U| + \sum_{b=0}^{d-1} \sum_{\alpha \in \{\pm(2^d+1)^{-1}\}} |U^c \cap \{M^{\mathcal{D}_{d,b,\alpha}} 1_U > 1 - \delta\}| \\ &\leq (1 + 2d[(1 - \delta)^{-1} - 1])|U|. \end{aligned}$$

The main line of the argument can now begin. We take  $\delta = (1 + 2^d)^{-1}$  for an integer  $d$ . We use the maximal functions  $M^{\mathcal{D}_d}$ , but only in the last step of the induction. Initialize  $\text{Enl}(n+1, \mathcal{U}') := \text{sh}(\mathcal{U}')$ , so that we use backwards induction. Inductively define

$$\begin{aligned} \text{Enl}(j, \mathcal{U}') &:= \{M_j^{\mathcal{D}} 1_{\text{Enl}(j+1, \mathcal{U}')} > 1 - \delta^{2^j}\}, \quad n \geq j \geq 2, \\ V &:= \{M_1^{\mathcal{D}_d} 1_{\text{Enl}(2, \mathcal{U}')} > 1 - \frac{\delta}{2}\}, \end{aligned} \tag{6.4}$$

where the subscript on the maximal functions denotes the coordinate in which the maximal function is applied. Then it is the case that  $|V| \leq (1 + K\delta|\log \delta|)|\text{sh}(\mathcal{U}')|$ , where the constant  $K$  depends only on the dimension  $n$ .

Now pass to a further subset  $\mathcal{U}'' \subset \mathcal{U}'$  such that for all  $R, R' \in \mathcal{U}''$ , we have  $2^{k-1} \leq \text{emb}(R, V) \leq 2^k$  and if it is the case that  $|R'_1| < |R_1|$ , then we have the stronger inequality  $40 \cdot 2^k \delta^{-1} |R'_1| < |R_1|$ . We term this assumption “separation of scales” in the first coordinate. Under these assumptions, estimate (6.2) reduces to

$$\sum_{I \in \mathcal{D}} |F(I, k, \mathcal{U}'')| \lesssim |\text{sh}(\mathcal{U}')|, \quad \mathcal{U}' \subset \mathcal{U}. \tag{6.5}$$

Sufficiency is seen by noting that obtaining separation of scales necessitates dividing the rectangles into approximately  $\log 2^k \delta^{-1}$  subclasses. Then multiplying by  $2^{-\epsilon k}$ , one is able to sum over all scales  $k$ .

Our strategy is to define, for dyadic intervals  $I$ , sets  $H(I)$  that are disjoint in  $I$ , contained in  $\text{sh}(\mathcal{U}')$ , and for which

$$1_{F(I, k, \mathcal{U}'')} \lesssim M 1_{H(I)}$$

for an appropriate maximal function  $M$ , where the implied constant is permitted to depend upon  $\delta$  and dimension  $n$ . It will in fact be of the order  $O(\delta^{-2^{n+1}})$ . An appeal to the Fefferman-Stein maximal inequalities [5] will then prove (6.5).

The sets  $H(I)$  are defined to be  $F(I, k, \mathcal{U}'') - G(I)$ , where

$$G(I) := \bigcup_{\substack{I' \in \mathcal{D} \\ I \subsetneq I'}} F(I', k, \mathcal{U}'').$$

By our separation of scales, the minimal dyadic interval  $\tilde{I}$  that contributes to this union contains  $I$  and satisfies

$$40 \cdot 2^k \delta^{-1} |I| < |\tilde{I}| \leq 80 \cdot 2^k \delta^{-1} |I|.$$

<sup>3</sup>The problem we are avoiding here is that the dyadic grid distinguishes dyadic rational points. At the point 0 for instance,  $((0, 1) - \delta) \not\subset (0, 2^k)$  for all integers  $k$ , regardless of how big  $k$  is.

<sup>4</sup>In fact, taking  $d = 1$ , it is routine to check that  $M^{\mathcal{D}_1}$  dominates an absolute multiple of the usual maximal function, thus, proving that it satisfies the weak type inequality.

We need to show that if  $R$  is a dyadic rectangle with  $|R \cap G(I)| > (1 - \delta^{2^{n+1}})|R|$ , then  $\text{emb}(R, V) \geq 2^{k+1}$ , and hence it can't be among those rectangles that contribute to  $F(I, k, \mathcal{U}'')$ .<sup>5</sup> This will be accomplished by the following device. We will show that

$$(6.6) \quad |\tilde{I} \times R_2 \times \cdots \times R_n \cap \text{Enl}(2, \mathcal{U}')| \geq (1 - \delta^{2^2})|\tilde{I} \times R_2 \times \cdots \times R_n|.$$

As  $\tilde{I} \pm \frac{\delta}{2}|\tilde{I}| \in \mathcal{D}_d$ , and this is the grid we use in the final step in the construction of  $V$ , we conclude that the rectangle  $\tilde{I} \times R_2 \times \cdots \times R_n$  is inside of  $V$ . Even  $\delta|\tilde{I}| > 20 \cdot 2^k|I|$ , therefore we see that  $\text{emb}(R, V) \geq 2^{k+1}$ , as desired.

We turn to the proof of (6.6). The sequence of powers of  $\delta$  that appear in the definition of  $V$ , (6.4), is explained in part by the next proposition.

**Proposition 6.7.** *Let  $0 \leq X \leq 1$  be a random variable on a probability space satisfying  $\mathbb{E}X = 1 - \eta$ . Then,*

$$\mathbb{P}(X < 1 - \sqrt{\eta}) \leq \sqrt{\eta}.$$

PROOF. Setting  $p = P(X < 1 - \sqrt{\eta})$ , the inequality

$$1 - \eta \leq \mathbb{E}X \leq (1 - \sqrt{\eta})p + 1 - p$$

will prove the proposition. □

We continue with the language of probability. Let  $(\Omega_j, \mathbb{P}_j)$  be the probability spaces

$$\begin{aligned} \Omega_1 &:= \tilde{I}, \\ \Omega_j &:= \tilde{I} \times R_2 \times \cdots \times R_j, \quad 2 \leq j \leq n, \end{aligned}$$

and let  $\mathbb{P}_j$  be normalized Lebesgue measure on  $\Omega_j$ . The first of the relevant sequence of random variables on these spaces is

$$\begin{aligned} X_{n-1}(x) &:= \frac{|(\{x\} \times R_n) \cap G(I)|}{|R_n|}, \quad x \in \Omega_{n-1}, \\ A_{n-1} &:= \{x \mid X_{n-1}(x) > 1 - \delta^{2^n}\}. \end{aligned}$$

Since  $|R \cap G(I)| > (1 - \delta^{2^{n+1}})|R|$ ,  $\mathbb{E}X_{n-1} \geq 1 - \delta^{2^{n+1}}$ , and applying the proposition,  $\mathbb{P}_{n-1}(A_{n-1}) > 1 - \delta^{2^n}$ . Continuing by reverse induction, define for  $n - 1 \geq j \geq 2$

$$\begin{aligned} X_{j-1}(x) &:= \frac{|(\{x\} \times R_j) \cap A_j|}{|R_j|}, \quad x \in \Omega_{j-1}, \\ A_{j-1} &:= \{x \mid X_{j-1}(x) > 1 - \delta^{2^j}\}. \end{aligned}$$

Induction gives us the conclusion that  $\mathbb{P}_1(A_1) > 1 - \delta^{2^2}$ . This implies (6.6) by inspection of definitions and so completes the proof.

**A Second Geometric Formulation.** We need a certain variant of the previous lemma. Given a set of rectangles  $\mathcal{U}$ , we let  $\text{emb}(\cdot) : \mathcal{U} \rightarrow [1, \infty)$  be a map from  $\mathcal{U}$  to the reals greater than one. And we take  $\iota : \mathcal{U} \rightarrow \{1, 2, \dots, n\}$  which is simply a choice of coordinates. Based on these two data, for any subset  $\mathcal{U}' \subset \mathcal{U}$  we set

$$(6.8) \quad F(I, k, m, \mathcal{U}') = \bigcup \{R \in \mathcal{U}' \mid 2^k \leq \text{emb}(R) < 2^{k+1}, \iota(R) = m, R_m = I\}.$$

---

<sup>5</sup>If we could use the strong maximal function to define embeddedness, this conclusion would be immediate. Our more subtle definition of embeddedness appears to force the more complicated argument that follows.

**Lemma 6.9.** *Fix  $\delta, \epsilon > 0$ . For any collection of rectangles  $\mathcal{U}$  with finite shadow, we can select  $V \supset \text{sh}(\mathcal{U})$ , and data  $\text{emb}(\cdot)$  and  $\iota$  so that the following conditions hold.*

$$\begin{aligned} |V| &\leq (1 + \delta)|\text{sh}(\mathcal{U})|, \\ \text{emb}(R)R &\subset V, \quad R \in \mathcal{U}, \\ \sum_{k=0}^{\infty} \sum_{m=1}^n \sum_{I \in \mathcal{D}} 2^{-(n+\epsilon)k} |F(I, k, m, \mathcal{U}')| &\lesssim |\text{sh}(\mathcal{U}')|. \end{aligned}$$

*The implied constant in the last line depends only on  $\delta, \epsilon$  and dimension.*

The essential points for us are that the set  $V$  is not much larger than the shadow of  $\mathcal{U}$ , and that the rectangles  $R \in \mathcal{U}$ , after a dilation *uniform in all coordinates* by the embeddedness quantity, is contained in the enlarged set  $V$ . We find that the embeddedness quantity in the last line requires a large negative power, but that is a completely harmless fact in the context of the application we have in mind.

Again, the fact that the dyadic intervals distinguish certain points causes some difficulties for us, and we appeal to the shifted dyadic intervals (6.3) of the previous subsection, though our needs are not so refined in the current context. Let  $\mathcal{S}$  be the union of the dyadic intervals with the two collections  $\mathcal{D}_{\pm} := \mathcal{D}_{1,0,\pm\frac{1}{3}}$ . For any interval  $I$  of the real line, we can find an interval  $J \in \mathcal{S}$  with  $I \subset J \subset 4I$ . Indeed, let  $J'$  be the maximal dyadic interval contained in  $4I$  with  $|I \cap J'| \geq \frac{1}{2}|I|$ . If  $I \subset J'$  we are done, so assume that this is not the case. We necessarily have  $|J'| \geq \frac{9}{4}|I|$ , so that one of the two intervals  $J' \pm \frac{1}{3}|J'|$  contains  $I$ . Both of these intervals are in  $\mathcal{S}$ , so we are done.

The method of proof requires that we apply Lemma 6.1, although we find it necessary to apply it both inductively and to a wide range of possible collections of rectangles. In fact, it is useful to us that this lemma applies not just to collections of a subset  $\mathcal{U} \subset \otimes_{j=1}^n \mathcal{S}$  such that the shadow of  $\mathcal{U}$  is of finite measure. It also applies to all possible subsets of  $\mathcal{U}$ .

We apply Lemma 6.1 to  $\mathcal{U}^0 := \mathcal{U}$ . Thus, we get a set  $V^1 \supset \text{sh}(\mathcal{U}^0)$ , with  $|V^1| \leq (1 + \delta)|\text{sh}(\mathcal{U}^0)|$ , so that for

$$\text{emb}^1(R, V^1) := \sup\{\mu \geq 1 \mid \mu R_1 \times R_2 \times \cdots \times R_n \subset V^1\}.$$

we have the conclusion of Lemma 6.1 holding. We then construct  $\mathcal{U}^1 \subset \otimes_{j=1}^n \mathcal{S}$ . Set

$$\begin{aligned} \mathcal{U}^1 &:= \{\gamma \times \otimes_{j=2}^n R_j \mid R \in \mathcal{U}, \gamma \in \mathcal{S}, \\ &\quad (R_1 \cup \tfrac{1}{4} \text{emb}^1(R, V^1) R_1) \subset \gamma \subset \text{emb}^1(R, V^1) R_1\}. \end{aligned}$$

The inductive stage of the construction is this. For  $2 \leq m \leq n$ , given  $\mathcal{U}^{m-1} \subset \otimes_{j=1}^n \mathcal{S}$ , we apply Lemma 6.1 to get a set  $V^m$  satisfying

$$V^m \supset \text{sh}(\mathcal{U}^{m-1}), \quad |V^m| \leq (1 + \delta)|\text{sh}(\mathcal{U}^{m-1})|.$$

The embedding function for rectangles  $R \in \mathcal{U}^{m-1}$  is

$$\begin{aligned} \text{emb}^m(R, V^m) &:= \sup\{\mu \geq 1 \mid R_1 \times \cdots \times R_{m-1} \times \mu R_m \\ &\quad \times R_{m+1} \times \cdots \times R_n \subset V^m\}. \end{aligned}$$

And the conclusion of (6.2) holds. The collection  $\mathcal{U}^m$  is then taken to consist of all rectangles of the form

$$\otimes_{j=1}^{m-1} R_j \times \gamma \times \otimes_{j=m+2}^n R_j$$

where  $R \in \mathcal{U}^{m-1}$  and  $\gamma \in \mathcal{S}$  satisfies

$$(R_m \cup \tfrac{1}{4} \text{emb}^m(R, V^m) R_m) \subset \gamma \subset \text{emb}^m(R, V^m) R_m.$$



To prove Lemma 6.9, we take  $V := V^n$ . It is the case that

$$\begin{aligned} |V^n| &\leq (1 + \delta) |\text{sh}(\mathcal{U}^{n-1})| \\ &\leq (1 + \delta) |V^{n-1}| \\ &\leq (1 + \delta)^n |\text{sh}(\mathcal{U})|. \end{aligned}$$

The definition of the embedding function is not so straight forward. It is taken to be

$$\text{emb}(R) = \frac{1}{16} \inf_{1 \leq m \leq n} \beta^m(R)$$

where  $\beta^m(\cdot)$  are inductively defined below. The function  $\iota(R)$  is taken to be the coordinate in which the infimum for the embedding function is achieved.

Set  $\beta^1(R) := \text{emb}^1(R, V^1)$ . In the inductive step, for  $2 \leq m \leq n$ , set  $\gamma_m(R) := \inf_{j < m} \beta^j(R)$ . For  $1 < \gamma < \gamma_m(R)$ , let

$$\beta_\gamma^m(R) := \text{emb}^m(\varphi_\gamma^m(R), V^m)$$

where  $\varphi_\gamma^m(R) \in \mathcal{U}^{m-1}$  is the rectangle with  $\varphi_\gamma^m(R)_j = R_j$  for  $j \geq m$ , and for  $1 \leq j < m$ ,  $\varphi_\gamma^m(R)_j$  is the element of  $\mathcal{S}$  of maximal length such that

$$(R_j \cup \frac{1}{4}\gamma R_j) \subset \varphi_\gamma^m(R)_j \subset \gamma R_j.$$

Now, take  $\bar{\gamma}$  to be the largest value of  $1 \leq \gamma \leq \gamma_m(R)$  for which we have the inequality  $\beta_\gamma^m(R) \geq \gamma$ . Let us see that this definition of  $\bar{\gamma}$  makes sense. This last inequality is strict for  $\gamma = 1$ , and as  $\gamma$  increases,  $\beta_\gamma^m(R)$  decreases, so  $\bar{\gamma}$  is a well defined quantity. Then define  $\beta^m(R) := \beta_{\bar{\gamma}}^m(R)$ , and for our use below, set  $\varphi^m(R) := \varphi_{\bar{\gamma}}^m(R)$ .

The choices above prove our lemma, as we show now. For each rectangle  $R \in \mathcal{U}$ , it is clear that  $\text{emb}(R)R \subset V$ . Take  $\mathcal{U}' \subset \mathcal{U}$ . If we consider the sets  $F(I, k, m, \mathcal{U}')$  as in (6.8), then, by Lemma 6.1 applied in the  $m$ th coordinate,

$$\sum_{I \in \mathcal{D}} |F(I, k, m, \mathcal{U}')| \leq 2^{\epsilon k} |\text{sh}(\varphi^m(\mathcal{U}'))|.$$

While we have a very good estimate for the shadow of  $\varphi^m(\mathcal{U})$ , a corresponding good estimate for an arbitrary subset  $\mathcal{U}'$  seems very difficult to obtain. But it is a consequence of our construction that the rectangle  $\varphi^m(R)$  is a rectangle which agrees with  $R$  in the coordinates  $j \geq m$  and, for coordinates  $1 \leq j < m$ , is expanded by at most  $32 \text{emb}(R) \leq 2^{k+6}$ . Hence, we have the estimate

$$\left| \bigcup \{ \varphi^m(R) \mid R \in \mathcal{U}' \} \right| \lesssim 2^{nk} |\text{sh}(\mathcal{U}')|.$$

This follows from the weak  $L^1$  bound for the maximal function in one dimension, applied in each coordinate separately. It is in this last step that we lose the large power of the embeddedness. Our proof is complete.

**The  $BMO_{-1}(\otimes_1^n \mathbb{C}_+)$  Formulation.** The form in which we apply the previous lemma is this.

**Lemma 6.10.** *Given a function  $b$  with finite  $BMO_{-1}(\otimes_1^n \mathbb{C}_+)$  norm and a collection of rectangles  $\mathcal{U}$  whose shadow has finite measure, the following construction is possible. For all  $\epsilon, \delta > 0$ , there is a set  $V \supset \text{sh}(\mathcal{U})$  with  $|V| \leq (1 + \delta) |\text{sh}(\mathcal{U})|$ . To each  $R \in \mathcal{U}$ , there is a quantity  $\text{emb}(R) \geq 1$  so that*

$$\begin{aligned} \text{emb}(R)R &\subset V, \quad R \in \mathcal{U}, \\ \left\| \sum_{R \in \mathcal{U}} \text{emb}(R)^{-(n+\epsilon)} \langle b, v_R \rangle v_R \right\|_{BMO(\otimes_1^n \mathbb{C}_+)} &\leq K_{\delta, \epsilon} \|b\|_{BMO_{-1}(\otimes_1^n \mathbb{C}_+)}. \end{aligned}$$

For the proof, we apply Lemma 6.9. To check the conclusion of the lemma, we take a subset  $\mathcal{U}' \subset \mathcal{U}$  consisting of rectangles with  $2^k \leq \text{emb}(R) < 2^{k+1}$ . We then have

$$\begin{aligned} \sum_{R \in \mathcal{U}'} |\langle b, v_R \rangle|^2 &\leq \|b\|_{BMO_{-1}(\otimes_1^n \mathbb{C}_+)}^2 \sum_{I \in \mathcal{D}} \sum_{m=1}^n |F(I, k, m, \mathcal{U}')| \\ &\lesssim \|b\|_{BMO_{-1}(\otimes_1^n \mathbb{C}_+)}^2 2^{(n+\epsilon)k} |\text{sh}(\mathcal{U}')|. \end{aligned}$$

This is all we need to prove Lemma 6.10.

## REFERENCES

- [1] Chang, Sun-Yung A., *Carleson measure on the bi-disc*, Ann. of Math. (2) **109**, (1979), 3, 613—620. [1](#)
- [2] Chang, Sun-Yung A., Fefferman, Robert, *Some recent developments in Fourier analysis and  $H^p$ -theory on product domains*, Bull. Amer. Math. Soc. (N.S.) **12**, (1985), 1, 1—43. [1](#), [4](#)
- [3] Chang, Sun-Yung A., Fefferman, Robert, *A continuous version of duality of  $H^1$  with BMO on the bidisc*, Ann. of Math. (2) **112**, (1980), 1, 179—201. [1](#)
- [4] Coifman, R. R., Rochberg, R., Weiss, Guido, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103**, (1976), 3, 611—635.
- [5] Fefferman, C., Stein, E. M., *Some maximal inequalities*, Amer. J. Math. **93**, (1971), 107—115. [14](#)
- [6] Fefferman, C., Stein, E. M.,  *$H^p$  spaces of several variables*, Acta Math. **129**, (1972), 3-4, 137—193.
- [7] Fefferman, Robert, *Bounded mean oscillation on the polydisk*, Ann. of Math. (2) **110**, (1979), 3, 395—406.
- [8] Ferguson, Sarah H., Lacey, Michael T., *A characterization of product BMO by commutators*, Acta Math. **189**, (2002), 2, 143—160. [1](#), [2](#)
- [9] Ferguson, Sarah H., Sadosky, Cora, *Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures*, J. Anal. Math. **81**, (2000), 239—267.
- [10] Journé, Jean-Lin, *A covering lemma for product spaces*, Proc. Amer. Math. Soc. **96**, (1986), 4, 593—598. [1](#), [2](#), [13](#)
- [11] book Meyer, Yves, Coifman, Ronald, *Wavelets*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, (1997), [3](#)
- [12] Nehari, Z., *On bounded bilinear forms*, Ann. of Math. (2) **65**, (1957), 153—162.
- [13] Pipher, Jill, *Journé's covering lemma and its extension to higher dimensions*, Duke Math. J. **53**, (1986), 3, 683—690. [1](#), [2](#), [13](#)

MICHAEL LACEY, SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332 USA  
*E-mail address:* `lacey@math.gatech.edu`

ERIN TERWILLEGER, DEPARTMENT OF MATHEMATICS, U-3009, UNIVERSITY OF CONNECTICUT, STORRS, CT 06269 USA  
*E-mail address:* `terwillegger@math.uconn.edu`